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Duclos, Pierre; Jensen, Arne

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by

Pierre Duclos and Arne Jensen

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DEPARTMENT OF MATHEMATICAL SCIENCES
AALBORG UNIVERSITY
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 99 40 80 80 ▪ Telefax: +45 98 15 81 29
URL: http://www.math.aau.dk

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On self-adjointness of singular Floquet Hamiltonians

Pierre Duclos‡ and Arne Jensen

Department of Mathematical Sciences, Aalborg University, Fr. Bajers Vej 7G, DK-9220 Aalborg Ø, Denmark

E-mail: `matarne@math.aau.dk`

Abstract. Schrödinger equations with time-dependent interactions are studied. We investigate how to define the Floquet Hamiltonian as a self-adjoint operator, when the interaction is singular in time or space. Using these results we establish the existence of a bounded propagator, by applying a result due to J. Howland.

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1. Introduction

We consider the Floquet operator approach to Schrödinger equations with time-dependent interactions. We describe a method for proving that the Floquet operator is self-adjoint, when the time-dependent interaction has some singularities. This method is based on results by Kato [1] and Howland [2, 3]. For a survey on time-periodic Schrödinger equations, we refer to [4] and references therein.

To understand why self-adjointness of the Floquet Hamiltonian is important, we recall some terminology and results concerning Schrödinger equations with time-dependent interactions.

Let $H(t)$ be a family of self-adjoint operators on $\mathcal{H}$. Consider the problem

$$\mathrm{i}\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = H(t)\psi(t), \quad \psi(s) = \psi_0. \tag{1}$$

Under some additional conditions the solution is given by

$$\psi(t) = U(t,s)\psi_0. \tag{2}$$

The family of bounded operators $U(t,s)$ is called the propagator. It has the properties

$$U(t,t) = 1 \quad \text{and} \quad U(t,r)U(r,s) = U(t,s). \tag{3}$$

In [2] a method for Schrödinger equations analogous to the procedure in classical mechanics for treating time-dependent interactions is introduced. It is based on the Floquet Hamiltonian. Let D denote the self-adjoint operator obtained from $-\mathrm{i}\partial_t$, acting on the Hilbert space $L^2(\mathbf{R})$, or acting on the Hilbert space $L^2(\mathbf{S}^1)$, in the latter case

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with periodic boundary conditions. The Floquet Hamiltonian is the operator formally given by

$$K = D \otimes 1 + \int^{\oplus} H(t)\mathrm{d}t$$

on $\mathcal{K} = L^2(\mathbf{R}) \otimes \mathcal{H} \cong L^2(\mathbf{R}, \mathcal{H}, \mathrm{d}t)$ or $\mathcal{K} = L^2(\mathbf{S}^1) \otimes \mathcal{H} \cong L^2(\mathbf{S}^1, \mathcal{H}, \mathrm{d}t)$.

The connection between the propagator and the Floquet Hamiltonian is

$$\mathrm{e}^{-\mathrm{i}\sigma K}\psi(t) = U(t, t-\sigma)\psi(t-\sigma), \tag{4}$$

see [2, (1.2)].

We are interested in obtaining a bounded propagator $U(t, s)$. Using [2] the first, and usually most difficult, step is to prove that the Floquet Hamiltonian is self-adjoint. Once this step has been accomplished, one more condition has to be satisfied.

We recall the results from [2]. For $\phi \in C_0^1(\mathbf{R})$ we denote the operator of multiplication with $\phi(t)$ on $\mathcal{K}$ by $M(\phi)$. We let $\dot{\phi} = \mathrm{d}\phi/\mathrm{d}t$. The result [2, Theorem 2] states that if we have $M(\phi)\,\mathrm{dom}\,K \subseteq \mathrm{dom}\,K$ and

$$KM(\phi) - M(\phi)K \subset \mathrm{i}M(\dot{\phi}) \tag{5}$$

for all $\phi \in C_0^1(\mathbf{R})$, then we get a bounded propagator $U(t, s)$ from (4), if additionally K is self-adjoint. In [2, Theorem 3] the commutator condition above is replaced by the same commutator condition for the resolvent $G(z) = (K - z)^{-1}$, i.e.

$$G(z)M(\phi) - M(\phi)G(z) = \mathrm{i}G(z)M(\dot{\phi})G(z) \tag{6}$$

for all z with $\mathrm{Im}\, z \neq 0$.

We now return to the question of self-adjointness of the Floquet Hamiltonian. We consider a perturbative framework. Let H_0 be a self-adjoint operator in a Hilbert space $\mathcal{H}$, the unperturbed operator. Then $K_0 = D \otimes 1 + 1 \otimes H_0$ is essentially self-adjoint on the domain $\mathrm{dom}\, D \otimes \mathrm{dom}\, H_0$.

The problem we consider concerns the self-adjointness of $K = K_0 + V$ on the domain $\mathrm{dom}\, K_0$, where V is a perturbation. If V is bounded relative to K_0 with relative bound less than one, then it is well known that K is self-adjoint on $\mathrm{dom}\, K_0$ and essentially self-adjoint on $\mathrm{dom}\, D \otimes \mathrm{dom}\, H_0$.

Problems arise, when one tries to define K for perturbations V that are not relatively bounded. Two classic examples are the formal perturbations

$$V(t, x) = f(t)\delta(x) \quad \text{and} \quad V(t, x) = \delta(t)g(x) \tag{7}$$

in the case, where $H_0 = -\Delta$ on $\mathcal{H} = L^2(\mathbf{R})$, and f and g are functions on $\mathbf{R}$. The perturbations are too singular to be covered by the usual existence theorems for bounded propagators, see the results and references in [4].

Let us briefly outline the contents of the paper. In section 2 we introduce the method of defining a self-adjoint operator based on factored perturbations and a factored second resolvent equation. We also give some simple examples of applications of the method. In section 3 we consider in detail kicked systems, and in particular the one kick model. In section 4 we summarize the results obtained on existence of bounded propagators, based on the results in the previous sections.

2. The Howland-Kato method

We discuss a method for defining a self-adjoint operator through its resolvent, see [5, VIII§1.1]. This method was used effectively in the fundamental paper [1]. We recall the result.

Proposition 2.1 ([5, VIII§1.1]). *Let $\mathcal{O} \subset \mathbf{C}$ be an open set. Assume that $R\colon \mathcal{O} \to \mathcal{B}(\mathcal{H})$ is a family of operators satisfying the first resolvent equation*

$$R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2), \quad z_1, z_2 \in \mathcal{O}.$$

Assume that there exists $z_0 \in \mathcal{O}$, such that $\ker R(z_0) = \{0\}$*. Then there exists a closed operator T on $\mathcal{H}$, such that $R(z) = (T - z)^{-1}$ for all $z \in \mathcal{O}$. If $\mathcal{O} = \overline{\mathcal{O}}$ and $R(z)^* = R(\overline{z})$ for all $z \in \mathcal{O}$, then T is self-adjoint.*

The problem with using this result is that it can be difficult to characterize the domain, other than as $\operatorname{dom} T = \operatorname{ran} R(z)$, $z \in \mathcal{O}$.

2.1. Main theorem

The following result is obtained from [2, §§3,4], which relies on the method in [1].

Let $K_0 = D \otimes 1 + 1 \otimes H_0$, as defined in section 1, and let

$$G_0(z) = (K_0 - z)^{-1}.$$

We then introduce the following assumption.

Assumption 2.2. Let $\mathcal{L}$ be a Hilbert space. Let $\mathcal{O} \subset \mathbf{C}$ be an open set, such that $\mathcal{O} = \overline{\mathcal{O}}$. Assume that there exist closed operators A and B from $\mathcal{H}$ to $\mathcal{L}$ with the following properties:

(i) $\operatorname{dom} K_0 \subset \operatorname{dom} A$ and $\operatorname{dom} K_0 \subset \operatorname{dom} B$.

(ii) There exists $c < 1$ such that

$$\|AG_0(z)B^*u\| \leq c\|u\| \tag{8}$$

for all $z \in \mathcal{O}$ and all $u \in \operatorname{dom} B^*$.

(iii) For all $u, v \in \operatorname{dom} A \cap \operatorname{dom} B$ we have

$$\langle Au, Bv \rangle = \langle Bu, Av \rangle. \tag{9}$$

Note that the assumptions imply that $AG_0(z)$ and $BG_0(z)$ are bounded operator, and that the operator $AG_0(z)B^*$ extends to a bounded operator on $\mathcal{L}$ with norm less than or equal to $c < 1$. This extension is denoted by $Q(z)$. The assumption implies that $1 + Q(z)$ is invertible for all $z \in \mathcal{O}$. One can replace the assumption (ii) by the assumption that $AG_0(z)B^*$ extends to a bounded operator on $\mathcal{L}$, denoted by $Q(z)$, and that $1 + Q(z)$ is invertible for all $z \in \mathcal{O}$.

Theorem 2.3 ([1]). *Let Assumption* 2.2 *be satisfied. Define for* $z \in \mathcal{O}$

$$G(z) = G_0(z) - (BG_0(z))^*(1 + Q(z))^{-1}AG_0(z). \tag{10}$$

Then there exists a self-adjoint operator K*, such that* $G(z) = (K - z)^{-1}$ *for all* $z \in \mathcal{O}$*.*

The papers [1] and [2] have some further assumptions on A and B, but they are not needed in order to define the operator K. Let us note that Assumption 2.2(iii) is needed to get a self-adjoint operator.

Remark 2.4. As stated in [1] the factorization technique, i.e. considering perturbations formally of the form B^*A, gives a natural additive structure on the set of perturbations. More precisely, assume that A_j and B_j satisfy the conditions in Assumption 2.2 with auxiliary Hilbert spaces $\mathcal{L}_j$, $j = 1, 2$. Then if we take $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, $A = A_1 \oplus A_2$, and $B = B_1 \oplus B_2$, we have $B^*A = B_1^*A_1 + B_2^*A_2$, and the conditions (i) and (iii) Assumption 2.2 are again satisfied. Condition (ii) must be imposed additionally. A sufficient condition is the existence of a $c < 1$ such that

$$\begin{aligned}&\|A_1G_0(z)B_1^*u_1\| + \|A_2G_0(z)B_1^*u_1\| \\ &\qquad + \|A_1G_0(z)B_2^*u_2\| + \|A_2G_0(z)B_2^*u_2\| \leq c(\|u_1\|^2 + \|u_2\|^2)^{1/2}\end{aligned}$$

for all $u_1 \in \operatorname{dom} B_1^*$, $u_2 \in \operatorname{dom} B_2^*$, and $z \in \mathcal{O}$.

2.2. Examples

We now give some examples showing what kind of singularity can be handled by Theorem 2.3.

2.2.1. Singularity in time We take a perturbation formally given as $V = f \otimes g$, where f is a real-valued function and g is a bounded self-adjoint operator on $\mathcal{H}$. We define $a(t) = |f(t)|^{1/2}$ and $b(t) = \operatorname{sign}(f(t))a(t)$. We then define

$$A = a \otimes g \quad \textit{and} \quad B = b \otimes 1$$

and take $\mathcal{L} = \mathcal{K}$. We have the following result.

Proposition 2.5. *Assume that* $f \in L^1(\mathbf{R})$*, and real-valued. Then Assumption* 2.2 *holds for the operators* A *and* B *defined above.*

Proof. The assumption implies that we have $a, b \in L^2(\mathbf{R})$. We take $\operatorname{dom} a = \{u \in L^2(\mathbf{R}) \,|\, au \in L^2(\mathbf{R})\}$, i.e. the maximal domain. Then a is self-adjoint in $L^2(\mathbf{R})$ on this domain. It follows that A is an essentially self-adjoint operator on $\operatorname{dom} a \otimes \mathcal{H}$. A similar argument applies to B. We write the spectral decomposition of H_0 as $H_0 = \int_{\mathbf{R}} E \mathrm{d}P_E$. Then we have for $\operatorname{Im} z \neq 0$

$$\begin{aligned}\|AG_0(z)\| &= \|\int_{\mathbf{R}} A(D + E - z)^{-1} \otimes 1 \mathrm{d}(1 \otimes P_E)\| \\ &= \|(1 \otimes g) \int_{\mathbf{R}} a(D + E - z)^{-1} \otimes 1 \mathrm{d}(1 \otimes P_E)\|\end{aligned}$$

$$\begin{aligned} &\leq \|g\| \sup_{E\in\sigma(H_0)} \|a(D+E-z)^{-1}\| \\ &= \|g\|\|a(D-z)^{-1}\|, \end{aligned} \tag{11}$$

since $a(D+E-z)^{-1} = \mathrm{e}^{-iEt}a(D-z)^{-1}\mathrm{e}^{iEt}$.

Define

$$F(t) = \int_0^t |f(s)|\mathrm{d}s$$

Since $f \in L^1(\mathbf{R})$, F is a bounded function. A simple integration by parts argument gives for $v \in C_0^\infty(\mathbf{R})$ the estimate

$$\|av\|^2 = \int_{\mathbf{R}} |f(s)|\overline{v(s)}v(s)\mathrm{d}s \leq 2\|F\|_{L^\infty}\|v\|\|v'\| \leq \|F\|_{L^\infty}(\|v\|^2 + \|Dv\|^2).$$

Since $C_0^\infty(\mathbf{R})$ is a core for D, it follows that $a(D-z)^{-1}$ is a bounded operator. Analogous arguments can be used for B. Thus part (i) of the Assumption 2.2 has been verified.

To verify part (ii) we show that there exists a $c_0 > 0$ and a $c < 1$, such that

$$\|g\|\|a(D+E-z)^{-1}b^*\| \leq c$$

for all $E \in \mathbf{R}$ and for all z satisfying $|\operatorname{Im} z| > c_0$.

We write $q = a(D+E-z)^{-1}b^*$ and recall that this operator has the integral kernel

$$q(t,s) = a(t)\chi_{(-\infty,t)}(s)\exp(\mathrm{i}(z-E)(t-s))\overline{b(s)}, \quad \operatorname{Im} z > 0.$$

Then we have

$$\int_{\mathbf{R}}\int_{\mathbf{R}} |q(t,s)|^2\mathrm{d}s\mathrm{d}t = \int_{\mathbf{R}} |a(t)|^2 \int_{-\infty}^t \exp(-2\operatorname{Im} z(t-s))|b(s)|^2\mathrm{d}s\mathrm{d}t \leq \|a\|^2\|b\|^2 = \|f\|_{L^1}^2.$$

Thus q is a Hilbert-Schmidt operator, and an application of the Lebesgue dominated convergence theorem shows that its Hilbert-Schmidt norm tends to zero as $\operatorname{Im} z \to \infty$. The same result holds for $\operatorname{Im} z < 0$.

It is easy to see that the condition (iii) in Assumption 2.2 is satisfied. If we take $\mathcal{O} = \{z \,|\, |\operatorname{Im} z| > c_0\}$ for c_0 sufficiently large, then all conditions in Assumption 2.2 are satisfied. □

We note that an analogous result holds for the periodic case. Here one assumes $f \in L^1(\mathbf{S}^1)$ and real-valued.

Furthermore, due to Remark 2.4, the result also holds for perturbations

$$V = \sum_{j=1}^N f_j \otimes g_j,$$

where $f_j \in L^1(\mathbf{R})$ or $f_j \in L^1(\mathbf{S}^1)$, real-valued, and g_j bounded and self-adjoint.

2.2.2. Singularity in space We take again a perturbation of the form $V = f \otimes g$. This time we assume that $f \in L^\infty(\mathbf{R})$ and that $g = y^*x$, where x and y are closed operators from $\mathcal{H}$ to another Hilbert space $\mathcal{M}$, such that $\operatorname{dom} H_0 \subset \operatorname{dom} x$, $\operatorname{dom} H_0 \subset y$, and such that

$$\|x(H_0 - z)^{-1}y^*u\| \le c\|u\| < 1 \tag{12}$$

for all $u \in \operatorname{dom} y^*$ and for all z satisfying $|\operatorname{Im} z| > c_0$ for some $c_0 > 0$. Using the spectral decomposition of D the arguments above can be repeated. To satisfy the symmetry condition (iii) in Assumption 2.2 we assume that f is real-valued, and that $\langle xu, yv\rangle = \langle yu, xv\rangle$ for all $u, v \in \operatorname{dom} x \cap \operatorname{dom} y$.

Let us give a few examples of g satisfying these conditions. First we take $\mathcal{H} = L^2(\mathbf{R})$ and $H_0 = -\Delta$. As g we take the Dirac delta δ. We can write $\delta = \tau^*\tau$, where $\tau\colon H^1(\mathbf{R}) \to \mathbf{C}$ is given by $\tau u = u(0)$. We take $\mathcal{M} = \mathbf{C}$. The operator $\tau(H_0 - z)^{-1}\tau^*$ is multiplication by the number $\mathrm{i}/(2\sqrt{z})$, such that (12) is satisfied for c_0 sufficiently large. The symmetry conditon is trivially satisfied.

We can also consider the derivative of the Dirac delta δ' as a perturbation. Thus we take $g = \delta'$ and $\mathcal{M} = \mathbf{C}$. This time we factor as $g = \tilde{\tau}^*\tilde{\tau}$, where

$$\tilde{\tau}\colon H^1((-\infty, 0]) \oplus H^1([0, \infty)) \to \mathbf{C}$$

is given by

$$\tilde{\tau}u = u(0_+) - u(0_-).$$

Also in this case the operator $\tilde{\tau}(H_0 - z)^{-1}\tilde{\tau}^*$ is multiplication by $\mathrm{i}/(2\sqrt{z})$ on $\mathcal{M}$.

We refer to [6] for further information on point interactions and extensive references to the literature.

3. Kicked systems

We now consider a class of problems often called kicked systems, where the perturbation is a point interaction in time. We continue to let $D = -\mathrm{i}\partial_t$ on $L^2(\mathbf{R})$, with domain the Sobolev space $H^1(\mathbf{R})$. We introduce the shorthand notation

$$H^1_+ = H^1([0, \infty)) \quad \text{and} \quad H^1_- = H^1((-\infty, 0]).$$

The space $H^1([0, \infty))$ is defined as the closure of $C_0^\infty([0, \infty))$ in $H^1(\mathbf{R})$. We also write $L^2_+ = L^2([0, \infty))$ and $L^2_- = L^2((-\infty, 0])$. In this case it does not matter, whether the point zero is included or not. Let $a \in \mathbf{C}$, $|a| = 1$. Define the operator

$$U_a\colon L^2_- \oplus L^2_+ \to L^2_- \oplus L^2_+ \quad \text{as} \quad U_a u_- \oplus u_+ = u_- \oplus a u_+.$$

Then U_a is a unitary operator. We then define an operator $D_{\kappa(a)}$ as follows.

$$\operatorname{dom} D_{\kappa(a)} = U_a H^1(\mathbf{R}) = \{u \in H^1_- \oplus H^1_+ \mid u(0_+) = a u(0_-)\}$$

and $D_{\kappa(a)}u = -\mathrm{i}(u' \oplus u')$. Stated differently, we have $D_{\kappa(a)} = U_a D U_a^{-1}$. It follows that $D_{\kappa(a)}$ is a self-adjoint operator.

We now use some results from [7]. We define the two linear maps

$$\tau_\pm\colon H^1_\pm \to \mathbf{C}, \quad \tau_\pm u = u(0_\pm).$$

Now let $u \in \operatorname{dom} D$ and $v \in \operatorname{dom} D_{\kappa(a)}$. A simple computation shows that

$$\begin{aligned}\langle u, D_{\kappa(a)}\rangle - \langle Du, v\rangle &= -\mathrm{i}\overline{u(0)}(v(0_-) - v(0_+)) \\ &= \mathrm{i}\frac{a-1}{a+1}\overline{u(0)}(v(0_-) + v(0_+)) \\ &= \mathrm{i}\frac{a-1}{a+1}\left(\overline{u(0_-)}v(0_-) + \overline{u(0_+)}v(0_+)\right),\end{aligned} \tag{13}$$

where we use that $u(0) = u(0_-) = u(0_+)$. We introduce the function $\kappa(a) = \mathrm{i}(a-1)/(a+1)$. Note that $\kappa(a)$ is real-valued for $|a| = 1$. For $a = -1$ we interpret the value as ∞. This computation allows us to formally write

$$D_{\kappa(a)} = D + \kappa(a)(\tau_-^*\tau_- + \tau_+^*\tau_+).$$

We will now write the resolvent of $D_{\kappa(a)}$ in the form (10), in order to give an example of the self-adjoint extension provided by the approach using (10).

We introduce the operator

$$T\colon H^1_- \oplus H^1_+ \to \mathbf{C}^2, \quad Tu = \begin{bmatrix} u(0_-) \\ u(0_+) \end{bmatrix}. \tag{14}$$

We write $G_{\kappa(a)}(z) = (D_{\kappa(a)} - z)^{-1}$ and $G_0(z) = (D - z)^{-1}$. In (13) we take for $\operatorname{Im} z \neq 0$ $u = G_0(\overline{z})u_1$ and $v = G_{\kappa(a)}(z)v_1$ for arbitrary $u_1, v_1 \in L^2(\mathbf{R})$. Thus we get

$$\langle G_0(\overline{z})u_1, v_1\rangle = \langle u_1, G_{\kappa(a)}(z)v_1\rangle + \kappa(a)\langle TG_0(\overline{z})u_1, TG_{\kappa(a)}(z)v_1\rangle.$$

Since u_1 and v_1 are arbitrary, we conclude that

$$G_{\kappa(a)}(z) = G_0(z) - \kappa(a)(TG_0(\overline{z}))^*TG_{\kappa(a)}(z). \tag{15}$$

We apply T on the left and rewrite to get

$$[1 + \kappa(a)T(TG_0(\overline{z}))^*]TG_{\kappa(a)}(z) = TG_0(z).$$

We will show below that the operator $1 + \kappa(a)T(TG_0(\overline{z}))^*$ is invertible. Thus using the resulting expression for $TG_{\kappa(a)}(z)$ in (15) we finally get

$$G_{\kappa(a)}(z) = G_0(z) - \kappa(a)(TG_0(\overline{z}))^*[1 + \kappa(a)T(TG_0(\overline{z}))^*]^{-1}TG_0(z). \tag{16}$$

Thus if we take

$$A = T, \quad B = \kappa(a)T, \quad \text{and} \quad \mathcal{L} = \mathbf{C}^2,$$

we have verified the formula (10).

The important obervation is the following: In (16) the resolvent obtained from the right hand side is the resolvent of the operator $D_{\kappa(a)}$ with an explicitly known action and domain, whereas the operator obtained from (10) via Theorem 2.3 is not known that explicitly.

We return to the question concerning the invertibility of the operator $1 + \kappa(a)T(TG_0(\overline{z}))^*$. We consider the case $\operatorname{Im} z > 0$. Using the explicit kernel of $G_0(z)$ a simple computation yields

$$(TG_0(\overline{z}))^* \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} (t) = \mathrm{i}\chi_{0,\infty)}(t)\mathrm{e}^{\mathrm{i}tz}(\zeta_1 + \zeta_2),$$

such that $T(TG_0(\overline{z}))^*$ is given by the matrix

$$\begin{bmatrix} 0 & 0 \\ \mathrm{i} & \mathrm{i} \end{bmatrix}.$$

Let

$$X = 1 + \kappa(a)T(TG_0(\overline{z}))^* = \begin{bmatrix} 1 & 0 \\ \mathrm{i}\kappa(a) & 1 + \mathrm{i}\kappa(a) \end{bmatrix} \tag{17}$$

This matrix is invertible for all a, since $\kappa(a)$ is real. We have

$$X^{-1} = \begin{bmatrix} 1 & 0 \\ \dfrac{-\mathrm{i}\kappa(a)}{1 + \mathrm{i}\kappa(a)} & \dfrac{1}{1 + \mathrm{i}\kappa(a)} \end{bmatrix}.$$

Using this result, we can simplify (16) to the following expression:

$$G_{\kappa(a)}(z) = G_0(z) - \frac{\kappa(a)}{1 + \mathrm{i}\kappa(a)}(TG_0(\overline{z}))^* TG_0(z). \tag{18}$$

We apply these results to the one kick model. We take as our Hilbert space

$$\mathcal{K} = L^2(\mathbf{R}, dt) \otimes L^2(\mathbf{R}, dx) \cong L^2(\mathbf{R}, L^2(\mathbf{R}, dt), dx).$$

Let $V(x)$ be a bounded real-valued function. We let

$$K_0 = D \otimes 1 + 1 \otimes H_0, \quad D = -\mathrm{i}\partial_t, \quad H_0 = -\partial_x^2.$$

and

$$K = \int_{\mathbf{R}}^{\oplus} D_{V(x)} dx + 1 \otimes H_0,$$

where we now have taken $\kappa(a) = V(x)$. The resolvent $G(z) = (K - z)^{-1}$ can be computed as above. We now let $G_0(z) = (K_0 - z)^{-1}$. The result can be stated as follows.

Theorem 3.1. *For the resolvent defined above we have for* $\operatorname{Im} z > 0$

$$G(z) = G_0(z) - (TG_0(\overline{z}))^* \frac{V}{1 + \mathrm{i}V} TG_0(z). \tag{19}$$

Proof. We continue to use the notation T introduced in (14). We then define $\mathcal{L} = L^2(\mathbf{R}, dx) \oplus L^2(\mathbf{R}, dx)$, and $A, B: \mathcal{K} \to \mathcal{L}$ by the closure of

$$A_1 = T \otimes 1 \quad \text{and} \quad B_1 = T \otimes V,$$

where we take

$$\operatorname{dom} A_1 = \operatorname{dom} B_1 = (H^1_- \oplus H^1_+) \otimes L^2(\mathbf{R}, dx)$$

With these definitions the computations above can be repeated and lead to the result stated in the theorem. $\square$

Remark 3.2. As shown above the matrix given by (17) is invertible. We note that the operator $T(TG_0(z))^*$ is not (quasi)nilpotent, which is the case for the perturbations considered in [2]. In [2] the author considers a multiplicative potential $q(t, x)$, which satisfies the following conditions. Let

$$v_p(t) = \left(\int_{\mathbf{R}^n} |q(t, x)|^p \mathrm{d}x \right)^{1/p}$$

for some $p > n/2$, such that $v_p \in L^{r-\varepsilon}(\mathbf{R}) \cap L^{r+\varepsilon}(\mathbf{R})$ for some $\varepsilon > 0$, where $r = 2p/(2p - n)$. For the case $p = \infty$ one assumes $v_\infty \in L^1(\mathbf{R}) \cap L^{1+\varepsilon}(\mathbf{R})$. Thus the conditions considered above and in section 2 allow a wider class of interactions. Note that we do not consider the question of existence and completeness of the wave operators, which is one of the main results in [2].

4. Bounded propagators

In section 2 we have obtained self-adjointness of the Floquet Hamiltonian, defined through using the Howland-Kato method, for the case $\mathcal{K} = L^2(\mathbf{R}) \otimes L^2(\mathbf{R})$ and formally

$$K = D \otimes 1 + 1 \otimes (-\Delta) + V,$$

where we have considered perturbations of the form

$$V = f \otimes g.$$

In Proposition 2.5 we have assumed $f \in L^1(\mathbf{R})$, real-valued, and g a bounded self-adjoint operator on $L^2(\mathbf{R})$. To use the results from [2] it remains to verify one of the commutator conditions (5) and (6). This is straightforward, so we omit the details.

In section 2.2.2 we have considered $f \in L^\infty(\mathbf{R})$, real-valued, and g a factored perturbation. The example included point interactions. Again, the commutator condition is straightforward to verify.

Thus in these cases the results in [2] associate a bounded propagator with the formal problem. What cannot be obtained in general with this approach is a definition of the family $H(t)$ as a self-adjoint family.

In section 3 we considered in particular the one kick model. Again, the commutator condition is straightforward to verify, so also for this model we get a bounded propagator.

Thus we have shown that it is possible to associate a bounded propagator with time-dependent singular perturbations of the free Schrödinger operator.

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References

[1] Kato T 1966 Wave operators and similarity for some non-selfadjoint operators *Math. Ann.* **162** 258–79

[2] Howland J 1974 Stationary scattering theory for time-dependent Hamiltonians *Math. Ann.* **207** 315-35

[3] Howland J 1979 Scattering theory for Hamiltonians periodic in time *Indiana Univ. Math. J.* **28** 471–94

[4] Yajima K 2004 Time-periodic Schrödinger equations *Topics in the theory of Schrödinger operators* (World Scientific, Singapore) ed H Araki and H Ezawa pp 9–69

[5] Kato T 1966 *Perturbation theory for linear operators* (Springer-Verlag, Berlin-Heidelberg-New York)

[6] Albeverio S, Gesztesy F, Høegh-Krohn R, and Holden H 2005 *Solvable models in quantum mechanics* (Second edition (with an appendix by P Exner) AMS Chelsea Publishing, Providence, RI)

[7] Kurasov P and Boman J 1998 Finite rank singular perturbations and distributions with discontinuous test functions *Proc. Amer. Math. Soc.* **126** 1673–83